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C^k-RIGIDITY FOR HYPERBOLIC FLOWS II

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ABSTRACT
In this note we prove the following result: Any conjugating homeomorphism between two geodesic flows for compact negatively curved compact C^∞ surfaces is necessarily C^∞. This extends a result of Feldman and Ornstein. We also discuss some related results for hyperbolic flows and diffeomorphisms.

0. Introduction

In a recent paper J. Feldman and D. Ornstein proved some very interesting results on conjugating maps between geodesic flows for compact surfaces of negative curvature [6]. In particular they showed that any homeomorphism which conjugates the flows is necessarily continuously differentiable. In [13] the present author produced a modest improvement in their result, to the effect that the isomorphism must be C^k where C^k is the smoothness of the horocycle foliation. In this note we start from the Feldman–Orstein result to show that the *homeomorphism is automatically of class* C^∞ (without any additional hypothesis on the smoothness of the horocycle foliation).

Feldman and Ornstein also proved a companion result for horocycle flows (giving a C^1 version of Ratner's rigidity theorem for constant negative curvature surfaces). Combining our result with their work we arrive at the following version: *For any two measurably isomorphic horocycle flows, the conjugating map must have a* C^∞ *version.*

The techniques we use are similar to those in our earlier paper [13]. The new ingredient in the present work is a result which we established in our recent

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analysis of the differential zeta function (Proposition 1 in the present paper). The basic idea, for the geodesic flow case, is to first introduce Markov Poincaré sections to reduce this to a discrete problem, and then to prove that the conjugating map is C^∞ along horocycles at every point (by choosing sections containing a given point, and a neighbourhood in one of the horocycles passing through it). Finally we can deduce that the conjugating map is C^∞, in the usual sense, from the fact that it is C^∞ along the foliations by using a result of de la Llave, Marco and Moriyan (Lemma 2 in the present note).

After writing this article de la Llave informed us that he and Marco had independently established results which include ours. However, their proof is in keeping with the general program in [5] and as such is significantly different from our own (except for our use of Lemma 2).

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1. Preliminaries

Let S be a compact C^∞ Riemannian surface with strictly negative curvature. Let M denote its unit tangent bundle. We write the associated geodesic flow as $\phi_t : M \to M$.

This flow is Anosov, i.e. there exists a continuous splitting $TM = E^0 \oplus E^u \oplus E^s$ into $D\phi_t$-invariant subbundles such that:

(a) E^0 is one-dimensional and tangent to the flow;

(b) there exists $C, \lambda > 0$ such that $\| D\phi_t(v) \| \leqq Ce^{-\lambda t} \| v \|$, $t \geqq 0$, $v \in E^s$; $\| D\phi_{-t}(v) \| \leqq Ce^{-\lambda t} \| v \|$, $t \geqq 0$, $v \in E^u$ (cf. [1]).

The manifold M is three-dimensional. The stable manifold through $x \in M$ is a one-dimensional C^∞ submanifold given by

$$V_x^{ss}(\phi) = \{ y \mid d(\phi_t x, \phi_t y) \to 0 \text{ as } t \to +\infty \}.$$

Collectively, these form the leaves of a foliation $\mathscr{F}^{ss}(\phi)$ for M of class $C^{1+\alpha}$, for any $0 < \alpha < 1$ [8, 10]. The unstable manfolds are defined similarly by replacing ϕ_t by ϕ_{-t}. These produce a foliation $\mathscr{F}^u(\phi)$ for M of class $C^{1+\alpha}$ whose leaves are C^∞ and one-dimensional.

Since the foliations are $C^{1+\alpha}$ this is also true of the positive function $\lambda^u : M \to \mathbf{R}$ defined by

$$\lambda^u(x) = \lim_{t \to 0} \frac{1}{t} \log | \operatorname{Jac}(\phi_t |_{E_x^u}) | .$$

We can reparametrize the flow ϕ by scaling the velocity by λ^u to derive a new flow $\psi_t : M \to M$. More precisely, we let $k(x, t) = \int_0^t \lambda^u(\phi_\omega x)d\omega$ and choose $l(x, t)$ such that

$$k(x, l(x, t)) = t = l(x, k(x, t)), \qquad \text{for all } x \in M, \quad t \in \mathbf{R}.$$

We define $\psi_t(x) = \phi_{l(x,t)}(x)$. Since λ^u is a $C^{1+\alpha}$ function, the flow ψ is a C^1 flow which has the same orbits as ϕ.

Anosov showed that ψ is again an Anosov flow [1]. Let the hyperbolic splitting for ψ be $TM = F^0 \oplus F^u \oplus F^s$, where

$$F^0 = E^0,$$
$$F_x^s = \{\xi + z(x, \xi)E_x^0 \mid \xi \in E_x^s\},$$
$$F_x^u = \{\xi + z(x, \xi)E_x^0 \mid \xi \in E_x^u\},$$

with

$$z(x, \xi) = -\lambda^u(x) \int_0^\infty \{(D\phi_t\xi)(\lambda^v)/[\lambda^u(\phi_t x)]^2\}dt. \tag{1.1}$$

(For a brief account of these ideas, we refer the reader to [12].)

Generally, the strong stable manifolds $V_x^{ss}(\psi)$ and strong unstable manifolds $V_x^{su}(\psi)$ for ψ are different from those for ϕ. Let $\mathscr{F}^{ss}(\psi)$, $\mathscr{F}^{su}(\psi)$ be the associated foliations. Since ψ is an Anosov flow, we know that these foliations are at least Hölder continuous, i.e. C^β for some $\beta > 0$ [1].

We observe that reparametrization doesn't change the weak stable and weak unstable manifolds. i.e.

$$\bigcup_{t \in \mathbf{R}} V_{\phi_t x}^{ss}(\phi) = \bigcup_{t \in \mathbf{R}} V_{\psi_t x}^{ss}(\psi) \quad \text{and} \quad \bigcup_{t \in \mathbf{R}} V_{\phi_t x}^{su}(\phi) = \bigcup_{t \in \mathbf{R}} V_{\phi_t x}^{su}(\psi).$$

We denote the foliations for which these are the leaves by $\mathscr{F}^{ws}$ and $\mathscr{F}^{wu}$. These are $C^{1+\alpha}$ foliations for which the individual leaves are C^∞ immersed and two-dimensional.

We can say more about $V_x^{su}(\psi)$. The function ϕ^u is only $C^{1+\alpha}$ on M, but along each individual leaf of $\mathscr{F}^{su}(\phi)$ or $\mathscr{F}^{wu}$ the function is C^∞. We can therefore deduce that the function $z(x, \xi)$ is also C^∞ when restricted to a leaf of $\mathscr{F}^{wu}$.

By considering (1.1), we can conclude that each strong unstable leaf $V_x^u(\psi)$ for ψ is a C^∞ manifold. However, we must remark that we cannot draw similar conclusions about the strong stable manifolds for ψ.

If we had reparametrized ϕ with respect to the function

$$\lambda^s(x) = \lim_{t\to 0} \frac{1}{t} \log |\operatorname{Jac}(D\phi_{-t}|_{E^s_x})|,$$

we would have a $C^{1+\alpha}$ flow χ, each of whose strong stable leaves $V^{ss}_x(\chi)$ is C^∞ (but here we cannot draw conclusions about the strong unstable manifolds $V^{su}_x(\chi)$).

At this stage we want to introduce Markov sections.

For small $\varepsilon > 0$, let

$$V^{ss}_x(\phi; \varepsilon) = \{y \in M \mid d(\phi_t x, \phi_t y) \leqq \varepsilon \ \ \forall t \geqq 0, d(\phi_t x, \phi_t y) \to 0\},$$

$$V^{su}_x(\phi; \varepsilon) = \{y \in M \mid d(\phi_{-t} x, \phi_{-t} y) \leqq \varepsilon \ \ \forall t \geqq 0, d(\phi_{-t} x, \phi_{-t} y) \to 0\}.$$

For small $\eta > 0$, $\exists \delta > 0$ such that for $d(x, y) \leqq \delta \ \exists$ unique $|t| < \eta$ with $V^{ss}_{\phi_t x}(\phi; \varepsilon) \cap V^{su}_y(\phi; \varepsilon) \neq 0$. Furthermore, this intersection is a single point, which we denote by $[x, y]$.

Given $x \in M$ and $S_i \subseteq V^{ss}_x(\phi; \delta)$, $U_i \subseteq V^{su}_x(\phi; \delta)$, the sets $T_i = [S_i, U_i]$ are two-dimensional $C^{1+\alpha}$ sections transverse to the flow. Each T_i contains both S_i and U_i and is foliated by strong stable manifolds for ϕ.

We can choose a family $\mathscr{T} = \{T_1, \ldots, T_k\}$ such that any ε-segment of any ϕ-orbit must intersect an element of $\mathscr{T}$. We can also assume the family $\mathscr{T}$ are Markovian (we refer to [2] for full definitions), i.e. if an orbit passes through the interiors of $T_{i_1}, \ldots, T_{i_n}$ (in sequence) and a second orbit passes through the interiors of $T_{i_n}, \ldots, T_{i_{n+m}}$ (in sequence) then there exists an orbit passing through the interiors of $T_{i_1}, \ldots, T_{i_{n+m}}$ (in sequence).

By distorting $\mathscr{T}$ along orbits of the flow, we can construct a new family $\mathscr{R} = \{R_1, \ldots, R_k\}$ of sections. Furthermore, we can arrange that each $R_i = \langle S'_i, U'_i \rangle$, $i = 1, \ldots, k$, where $\langle \ , \ \rangle$ corresponds to the reparametrized flow ψ and S'_i, U'_i are pieces of strong stable and strong unstable manifolds for ψ.

Again, each R_i contains both S'_i and U'_i and is foliated by ψ strong stable manifolds. (The family $\mathscr{R}$ is automatically Markov since $\mathscr{T}$ is Markov.)

Let $\pi : \coprod_i R_i \to \coprod_i R_i$ be the Poincaré map on the sections $\mathscr{R}$. By identifying points on the same strong stable manifold, we can consider the induced map $f: J \to J$ where $J = \coprod_i U'_i$. The map J is uniformly expanding, Markov and (without additional assumptions) has the same degree of differentiability as the foliations, i.e. C^β.

We can make the additional assumption that the sections $\mathscr{R}$ are in *preferred form*, in the following sense:

(a) $\operatorname{Card}\{f^{-1}(x)\} \leqq 2$.

(b) If $f(y_1) = f(y_2) = x$, then either $\pi(y_1) = x$ or $\pi(y_2) = x$.

(c) If $f^{-1}(x) = \{y\}$, then $\pi(y) = x$.

A full account of constructing sections in preferred form is given in [14].

Let m be the measure of maximal entropy for $\psi_t : M \to M$, i.e. $h(m) = h(\psi)$ where $h(\psi)$ denotes the topological entropy of ψ. Simple considerations show that $h(\psi) = 1$ (cf. [12], for example).

From work of Margulis, the measure m is known to induce a unique transverse measure $\hat{m} = \{m_D\}_D$ for the foliation $\mathscr{F}^{ss}(\psi)$. (Here D runs through all compact transverse sections.) Furthermore, $\psi_t^* m_D = e^{h(\psi)t} m_{\psi_t D}$ [11].

The flow ψ is *synchronized* (in the sense of Parry [12]), i.e. the measure of maximal entropy is coincident with the unique smooth ψ-invariant measure on M. Since m is smooth on M, the induced measure m_D on each transverse section D is also smooth.

We shall concentrate on those sections D supported in single leaves of $\mathscr{F}^{wu}$. Since the action of ψ uniformly expands these sections, the measure m_D has a C^∞ smooth density with respect to the induced Riemannian metric (cf. [14] for details and [5] for related work). In consequence, the induced measure μ on the C^∞ one-dimensional manifold J has a C^∞ smooth density with respect to the induced Riemannian metric, i.e. $d\mu = \rho dx$, where $\rho : J \to \mathbf{R}^+$ is C^∞.

In the Bowen–Marcus construction of the transverse measure $\hat{m}$, they first construct a measure like μ (without necessarily using sections in preferred form) and then use this to reconstruct $\hat{m}$. In particular, their analysis gives that ρ is an eigenfunction for an operator on $C^\infty(J)$ and satisfies

$$\rho(x) = \rho(y_1)\mathrm{Jac}(f)(y_1) + \rho(y_2)\mathrm{Jac}(f)(y_2). \tag{1.2}$$

Since the sections are assumed in preferred form, we can assume that for y_1 (after interchanging the labeling, if necessary) the function Jac(f)(y_1) has a C^∞ dependence on $x \in J$. The above identity (1.2) allows us to reach the same conclusion about $\mathrm{Jac}(f)(y_2)$. Since J is one-dimensional, information on the Jacobian gives information about derivatives. We summarize:

PROPOSITION 1 [14]. *For the synchronized flow ψ and the family of Markov sections in preferred form, the induced transformation $f: J \to J$ is C^∞.*

2. C^∞-rigidity

Assume that S_1 and S_2 are two compact C^∞-Riemannian manifolds with negative curvature and let $\phi_t^1 : M_1 \to M_1$ and $\phi_t^2 : M_2 \to M_2$ be the associated geodesic flows. The following elegant result was proved by J. Feldman and D. Ornstein [6]:

PROPOSITION 2 (Feldman–Ornstein). *If $h : M_1 \to M_2$ is a homeomorphism which conjugates the flows ϕ^1 and ϕ^2, then h is C^1.*

The present author showed that under the (very restrictive) assumption that the horocycle foliations are C^∞, the map h must have a C^∞ version [7], even for the more general case of Anosov flows. (It was shown by Ghys and Hurder–Katok that for geodesic flows, as soon as the foliations are even C^2, then the surfaces are of constant curvature [7], [10]).

THEOREM 1. *Let h be a homeomorphism which conjugates the geodesic flows ϕ^1 and ϕ^2 for compact surfaces of negative curvature; then h is necessarily of class C^∞.*

PROOF. We can immediately invoke the original result of Feldman and Ornstein (Proposition 2) to deduce that h is C^1. We now need to adapt the techniques in [13] to show that h has a C^∞ version.

We let

$$\lambda^{1,u}(x) = \lim_{t \to 0} \frac{1}{t} \log |\mathrm{Jac}(\phi^1_{-t} |_{E^s_x})|$$

and reparametrize ϕ^1 by this $C^{1+\alpha}$ function to get a new (synchronized) flow $\psi^1 : M_1 \to M_1$. Similarly, we let

$$\lambda^{2,u}(x) = \lim_{t \to 0} \frac{1}{t} \log |\mathrm{Jac}(\phi^2_{-t} |_{E^s_x})|$$

and reparametrize ϕ^2 by this $C^{1+\alpha}$ function to get a (synchronized) flow $\psi^2 : M_2 \to M_2$. As explained in section 1, the flows ψ^1, ψ^2 will have the property that their measures of maximal entropy will be smooth on the unit tangent bundles M_1, M_2, respectively.

We can construct Markov sections $\mathcal{T}^1 = \{T^1_1, \ldots, T^1_k\}$ for $\phi^1_t : M_1 \to M_1$ which are in preferred form.

Since the flows ϕ^1 and ϕ^2 are topologically conjugate by $h : M_1 \to M_2$, we can deduce that h takes the foliation $\mathcal{F}^{ss}(\phi^1)$ to $\mathcal{F}^{ss}(\phi^2)$ and similarly takes the foliation $\mathcal{F}^{su}(\phi^1)$ to $\mathcal{F}^{su}(\phi^2)$. Thus we can define a family of Markov sections $\mathcal{T}^2 = \{T^2_1, \ldots, T^2_k\}$ for $\phi^2_t : M_2 \to M_2$ by $T^2_i = h(T^1_i)$, $1 \leqq i \leqq k$. The family $\mathcal{T}^2$ will again have preferred form.

By distorting $\mathcal{T}^1$ and $\mathcal{T}^2$ along the orbits of the flows ϕ^1 and ϕ^2, respectively, we can construct Markov sections $\mathcal{R}^1 = \{R_1^1, \ldots, R_k^1\}$ for $\psi_t^1 : M_1 \to M_1$ and $\mathcal{R}^2 = \{R_1^2, \ldots, R_K^2\}$ for $\psi_t^2 : M_2 \to M_2$, which are in preferred form with respect to these synchronized flows ψ^1 and ψ^2.

We can use the notation of the preceding section to write $T_i^1 = [S_i^1, U_i^1]$, $T_i^2 = [S_i^2, U_i^2]$, and by construction we can write $S_i^2 = h(S_i^1)$, $U_i^2 = h(U_i^1)$, $1 \leqq i \leqq k$. We can repeat this for the partitions $\mathcal{R}^1$ and $\mathcal{R}^2$ and write $R_i^1 = \langle \tilde{S}_i^1, \tilde{U}_i^1 \rangle$, $R_i^2 = \langle \tilde{S}_i^2, \tilde{U}_i^2 \rangle$, $1 \leqq i \leqq k$.

The map $h : M_1 \to M_2$ restricts to a map $H : T_i^1 \to T_i^2$ on each of the sections $(i = 1, \ldots, k)$. By identifying along strong stable manifolds, this induces a map $h : U_i^1 \to U_i^2$ $(i = 1, \ldots, k)$.

We prefer to work with the sections $\mathcal{T}^1$, $\mathcal{T}^2$. To this end, we introduce the maps $\pi_i^1 : \tilde{U}_i^1 \to U_i^1$, $\pi_i^2 : \tilde{U}_i^2 \to U_i^2$, $i = 1, \ldots, k$, as the projection along flow lines. We denote

$$\tilde{\mathcal{J}}^1 = \coprod_{i=1}^{k} \tilde{U}_i^1 \quad \text{and} \quad \tilde{\mathcal{J}}^2 = \coprod_{i=1}^{k} \tilde{U}_i^2$$

and define a map $\hat{h} : \tilde{\mathcal{J}}^1 \to \tilde{\mathcal{J}}^2$ by $\hat{h}(x) = (\pi_i^2)^{-1} h \pi_j^1(x)$ for $x \in \tilde{\mathcal{J}}_j^1 \cap \pi^{-1} \tilde{\mathcal{J}}_i^1$.

The Poincaré map for $\psi_t^1 : M_1 \to M_1$ on $\mathcal{R}^1$ induces a map $f_1 : \tilde{\mathcal{J}}^1 \to \tilde{\mathcal{J}}^1$ which is uniformly expanding and Markov. Since ψ^1 is synchronized and $\mathcal{R}^1$ are in preferred form, we can use Proposition 1 to conclude that f_1 is C^∞. Similarly, the Poincaré map for $\psi^2 : M_2 \to M_2$ on $\mathcal{R}^2$ induces a (piecewise) C^∞ uniformly expanding Markov map, denoted f_2.

By construction the map $\hat{h} : \mathcal{T}^1 \to \mathcal{T}^2$ is of class C^1. Furthermore, $\hat{h}$ conjugates $f_1 : \mathcal{T}^1 \to \mathcal{T}^1$ and $f_2 : \mathcal{T}^2 \to \mathcal{T}^2$, i.e. $f_2 \hat{h} = \hat{h} f_1$.

We want to invoke the following (slight) generalization of a result of Shub and Sullivan:

LEMMA 1 (cf. [13]). *If $f_1 : \tilde{\mathcal{J}}^1 \to \tilde{\mathcal{J}}^1$, $f_2 : \tilde{\mathcal{J}}^2 \to \tilde{\mathcal{J}}^2$ are piecewise C^∞ uniformly expanding Markov interval maps and $\hat{h} : \tilde{\mathcal{J}}^1 \to \tilde{\mathcal{J}}^2$ is a C^1 conjugating map, then $\hat{h}$ is necessarily of class C^∞.*

The maps π_i^1, π_i^2, $1 \leqq i \leqq k$, are C^∞ by construction. Thus the definition of $\hat{h}$ and Lemma 1 show that $h : U_i^1 \to U_i^2$ is C^∞ for $1 \leqq i \leqq k$.

We can conclude that $h : M_1 \to M_2$ is C^∞ *along* strong unstable manifolds $V_x^{su}(\phi^1)$, for any $x \in M_1$. To see this we need only arrange that our Markov sections $\mathcal{T}_1$ (in preferred form) have $x \in \operatorname{int} U_i^1$, for some $1 \leqq i \leqq k$. The above analysis tells us $h : M_1 \to M_2$ is C^∞ at $x \in U_i^1 \subseteq M$ along $U_i^1 \subseteq V_x^{ss}(\phi^1)$. Since smoothness of h along leaves of $\mathscr{F}^{ss}(\phi^1)$ is a local property, this is sufficient to prove this result.

The above argument can be easily adapted to show $h : M_1 \to M_2$ is C^∞ along strong stable manifolds $V_x^{ss}(\phi^1)$, for any $x \in M_1$. (All that is required is to reverse the time and define $\tilde{\phi}_t^1 : M_1 \to M_1$ by $\tilde{\phi}_t^1 = \phi_{-t}^1$ and $\tilde{\phi}_t^2 : M_2 \to M_2$ by $\tilde{\phi}_t^2 = \phi_{-t}^2$.) This has the effect that $V_x^{su}(\tilde{\phi}^1) = V_x^{ss}(\phi^1)$ (and $V_x^{su}(\tilde{\phi}^2) = V_x^{ss}(\phi^2)$). By replacing ϕ^1 by $\tilde{\phi}^1$ (and ϕ^2 by $\tilde{\phi}^2$) in the above analysis we conclude that $h : M_1 \to M_2$ is C^∞ along the leaves of $\mathscr{F}^{su}(\tilde{\phi}^1)$, i.e. along the leaves of $\mathscr{F}^{ss}(\phi^1)$.

Let $\mathscr{F}_1 = \mathscr{F}^{ss}(\phi^1)$ and $\mathscr{F}_2 = \mathscr{F}^{wu}(\phi^1)$. Then $h : M_1 \to M_2$ is C^∞ *along* the leaves of each of these foliations.

The conjugating map $\hat{h} : \tilde{\mathscr{J}}^1 \to \tilde{\mathscr{J}}^2$ has derivatives $\hat{h}^{(k)}$, $k \geqq 0$, by Lemma 1. These are determined by derivatives for f_1, f_2 and the density of the invariant measure via Lemma 1 (cf. [13]).

We can then see that higher derivatives of $\hat{h}$ and h vary continuously over M_1.

Let $g : M_2 \to \mathbf{R}$ be a C^∞ function and let $k = g \circ h : M_1 \to \mathbf{R}$. Clearly k is C^∞ *along* the leaves of the two foliations, and the derivatives vary continuously over M_1.

We require the following result due to de la Llave, Marco and Moriyon [5] (cf. also Katok–Hurder [10] and Journé [9] for alternative proofs).

Lemma 2 (cf. [5]). *If $k : M_1 \to \mathbf{R}$ is C^∞* along *the leaves of both foliations $\mathscr{F}_1$ and $\mathscr{F}_2$ (which satisfy a suitable regularity condition) and all the derivatives vary continuously on M_1, then k is C^∞ in the usual sense.*

We conclude that $k = g \circ h : M_1 \to \mathbf{R}$ is C^∞. Since we have complete freedom in choosing $g \in C^\infty(M_2, \mathbf{R})$, we can conclude that $h : M_1 \to M_2$ is C^∞. [This is clearer if we move to $\mathbf{R}^3$, using smooth charts, where smoothness is equivalent to smoothness of the one-dimensional projections.]

The original Feldman–Ornstein result included a formulation for horocycle flows (for surfaces of [variable] negative curvature) in analogy with Ratner's well-known horocycle rigidity theorem. By combining Theorem 1 with the horocycle flow result in [6], we get a C^∞ version of Ratner's theorem.

Corollary 1.1. *Let two horocycle flows for C^∞ compact surfaces of (variable) negative curvature be conjugate via a measurable map h which is absolutely continuous with respect to smooth measures. The map h has a C^∞ version.*

We should make it clear that: (a) the flow in question is the horocycle foliation, so parametrized (by Margulis) that the parameter is uniformly expanded by the geodesic flow; (b) the measurability is with respect to the measure of maximal entropy for the geodesic flow; (c) "has a C^∞ version" means "agrees with a C^∞ function on a set of zero measure with respect to the measure of maximal entropy for the geodesic flow".

The *length spectrum* of a C^∞ surface is a function which associates to each free homotopy class the length of the unique closed geodesic it contains.

Combining Theorem 1 with (10.3) in [4], we get:

COROLLARY 1.2. *Let two C^∞ compact surfaces of (variable) negative curvature have the same length spectrum. Then the associated geodesic flows are C^∞ conjugate.*

REMARK. After this paper was written J.-P. Otal proved the far more complete result that under the above hypothesis of Corollary 1.2 the surfaces are, in fact, isometric.

It is a simple matter to modify the above results to apply to more general settings. Since the Feldman and Ornstein proved Proposition 2 for the more general case of arbitrary C^∞ weak-mixing transitive Anosov flows on three-dimensional flows whose horocycle foliations are C^1 (termed "Good" flows), the proof of Theorem 1 immediately gives: *Let h be a homeomorphism that conjugates two good flows; then h is necessarily of class C^∞.*

The hypothesis that the horocycle foliation be C^1 is somewhat restrictive for general Anosov flows on three-dimensional manifolds. The only place we really needed it was in invoking the result of Feldman and Ornstein, which in turn shows that the conjugating homeomorphism h preserved the two SRB-measures (since this is automatically true when h is a diffeomorphism, which is guaranteed by Feldman–Ornstein). If we by-pass this by the assumption that h preserves the SRB-measures, then this implies that the map h is absolutely continuous. However, Lemma 1 remains true under this weakened hypothesis, cf. [13], and we can ultimately conclude that h is of class C^∞. Therefore we need only to assume only that ϕ^1 and ϕ^2 are arbitrary C^∞ transitive Anosov flows on three-dimensional manifolds, *provided* we make the stronger hypothesis on the conjugating map h that it should preserve the two SRB-measures (or, what is equivalent, that under h should have the same "exponents" $\int_\tau \lambda^{i,u} dt$ by integrating $\lambda^{i,u}$ ($i = 1, 2$) over corresponding closed orbits τ_1 and τ_2 for ϕ^1 and ϕ^2, respectively). In conclusion: *Let h be a homeomorphism which conjugates two C^∞ transitive three-dimensional Anosov flows and preserves the SRB-*

measures (or equivalently preserves the exponents of corresponding closed orbits), then h is necessarily of class C^∞.

REMARKS. (1) In Theorem 1 we can assume that the surfaces are C^k, $k \geqq 3$; then the associated geodesic flows are of class C^{k-1} and the foliations are still $C^{1+\alpha}$. The proof still carries through to show that any absolutely continuous conjugacy must have a C^r version ($r = r(k) \geqq 0$ can be deduced from the Sobolov estimates in [5]).

(2) We have concerned ourselves exclusively with the case of flows. However, as might be supposed, some of the above results carry over to two-dimensional transitive Anosov diffeomorphisms by taking suspensions. In particular, we immediately have: *Let h be a homeomorphism that conjugates two* C^∞ *transitive two-dimensional Anosov diffeomorphisms and preserves both SRB-measures (or equivalently preserves the "exponents" of corresponding closed orbits), then h is necessarily of class* C^∞.

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